

HOW ARE BLACK HOLES QUANTIZED?

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Abstract

Some approaches to quantization of the horizon area of black holes are discussed. The maximum entropy of a quantized surface is demonstrated to be proportional to the surface area in the classical limit. This result is valid for a rather general class of approaches to surface quantization. In the case of rotating black holes no satisfactory solution for the quantization problem has been found up to now.

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1 Introduction

Some properties of black holes, both classical and quantum, were discussed at XXXII Winter School of PNPI [1]. The last part of that talk referred to the quantization of black holes. Here we will consider this problem from somewhat different point of view. But for the completeness sake let us recall at first some considerations from [1].

The idea of quantizing the horizon area of black holes was proposed by Bekenstein many years ago [2]. It was based on the fact that the horizon area of a nonextremal black hole behaves in a sense as an adiabatic invariant [3]. And the quantization of an adiabatic invariant looks perfectly natural. Once this hypothesis is accepted, the general structure of the quantization condition for large quantum numbers gets obvious up to an overall numerical constant α (our argument here goes back to [4] and somewhat differs from that of the original one [2]). The quantization condition for the horizon area A should be

$$A = \alpha l_p^2 N, \quad (1)$$

where N is some large quantum number which generally speaking should not be an integer. Indeed, the presence of the Planck length squared

$$l_p^2 = k\hbar/c^3 \quad (2)$$

(here k is the Newton gravitational constant) is only natural in this quantization rule. Then, for the horizon area A to be finite in the classical limit, the power of N here should be the same as that of \hbar in l_p^2 .

From different points of view the black hole quantization was discussed later in Refs. [5, 6]. In particular, the article by Kogan [6] was the first investigation of the problem in the frame of the very popular now string approach. However, though a lot of work has been done since on the subject, especially based on the string theory and on the loop quantum gravity, I do not think that the problem is really solved now. Some attempts to attack the problem of black hole quantization are described below.

2 "It from bit"

The quantization condition (1) suggests that the horizon surface splits into patches of the area $\sim l_p^2$ each. But what can one say about the contribution by each of them into the total area?

Perhaps, the simplest model of the quantization, naturally called "it from bit", is due to Wheeler [7]. Here the horizon surface consists of ν patches of area $\sim l_p^2$ (of course, for a classical black hole $\nu \gg 1$), and to each of them one assigns a spin variable with two possible values $j_z = +1/2$ and $j_z = -1/2$. Thus, the total number of states is $K = 2^\nu$. The black hole entropy defined as log of number of states is

$$S = \ln K = \nu \ln 2. \quad (3)$$

On the other hand, the entropy of a black hole is related to the area A of its horizon by the famous Bekenstein-Hawking relation (see, e.g., [1])

$$A = 4l_p^2 S. \quad (4)$$

In this way one arrives at the quantization rule for the horizon area

$$A = 4 \ln 2 l_p^2 \nu, \quad (5)$$

with an integer ν and fixed overall numerical constant $\alpha = 4 \ln 2$.

It is absolutely crucial here that the patches should be distinguishable. Otherwise, instead of 2^ν states, we would have only $\nu + 1$ states: a single one with all spins up, a single one with one spin down and all other spins up, a single one with two spins down and all other spins up, and so on. Thus, with indistinguishable patches the entropy would be $S = \ln(\nu + 1)$, which has nothing in common with the required proportionality to the area $A \sim l_p^2 \nu$, especially if one recalls the classical condition $\nu \gg 1$.

One more point should be mentioned here. With the spins assigned to the patches being the only building blocks of the model, it will be natural to assume that the total angular momentum J of the black hole is just the sum of these spins. It is clear intuitively that for the random distribution of j_z over the patches this total angular momentum should be small. To find how small exactly, let us calculate at first the number $K(J_z = 0)$ of states with $J_z = \sum j_z = 0$. Evidently, here the number ν_\pm of patches with spins up and down is

$$\nu_+ = \nu_- = \nu/2 \quad (\nu \gg 1).$$

Thus,

$$K(J_z = 0) = \frac{\nu!}{[(\nu/2)!]^2}.$$

And the entropy of this state, as estimated for $\nu \gg 1$ with the Stirling formula,

$$S(J_z = 0) = \ln K(J_z = 0) = \nu \ln 2 - \frac{1}{2} \ln \nu \quad (6)$$

indeed differs from the initial one (3) by a logarithmic correction only.

As to the number of states with $J = 0$, $K(J = 0)$, it equals obviously to the difference between the numbers of the states with $J_z = 0$ and $J_z = \pm 1$:

$$K(J = 0) = K(J_z = 0) - K(J_z = \pm 1) = \frac{\nu!}{[(\nu/2)!]^2} - \frac{\nu!}{(\nu/2 - 1)!(\nu/2 + 1)!} = \frac{\nu!}{(\nu/2)!(\nu/2 + 1)!}.$$

The entropy of this state, with $J = 0$,

$$S(J = 0) = \ln K(J = 0) = \nu \ln 2 - \frac{3}{2} \ln \nu \quad (7)$$

again differs from (3) (and from (6) as well) by a logarithmic correction only.

Thus, even without the subsidiary condition $J = 0$, the simple-minded "it from bit" model can describe in a rather natural way the Schwarzschild black hole.

Let us go over now to the Kerr black hole, rotating one. Its entropy looks as follows (see, e.g., [1]):

$$S = 2\pi \left(M^2 + \sqrt{M^4 - J^2} \right); \quad (8)$$

here M is the black hole mass (to simplify formulae we put now $c = 1$, $\hbar = 1$, $k = 1$). As to the black hole angular momentum J , we assume again that it is the sum of the spins of the patches: $J = \sum \mathbf{j}$. Let us see whether the "it from bit" model can reproduce in this way the semiclassical relation (8). It can be easily demonstrated that in the discussed model the number of states with total angular momentum J equals

$$\begin{aligned} K(J) &= (2J + 1) [K(J_z = J) - K(J_z = J + 1)] \\ &= \nu!(2J + 1) \left[\frac{1}{(\nu/2 - J)!(\nu/2 + J)!} - \frac{1}{(\nu/2 - J - 1)!(\nu/2 + J + 1)!} \right] \end{aligned} \quad (9)$$

$$= \frac{\nu!(2J+1)^2}{(\nu/2-J)!(\nu/2+J+1)!}$$

In the limiting case $1 \ll J \ll \nu$ the "it from bit" entropy is

$$S_{\text{ifb}} = \ln K(J) = \nu \ln 2 - \frac{2J^2}{\nu}. \quad (10)$$

But let us look now at the corresponding expansion of the semiclassical formula (8):

$$S = 2\pi \left(2M^2 - \frac{J^2}{2M^2} \right). \quad (11)$$

Obviously, we should identify $2M^2$ with $\nu \ln 2 / 2\pi$. But then formula (11) is rewritten as

$$S = \nu \ln 2 - \frac{(2\pi)^2}{\ln 2} \frac{J^2}{\nu}. \quad (12)$$

The difference between numerical factors at J^2/ν is tremendous, the "it from bit" model strongly underestimates the entropy decrease with J in the discussed regime.

In the limit of the extremal black hole, where according to the semiclassical formula (8) J acquires its maximum value M^2 and the entropy equals $2\pi M^2$, the situation with "it from bit" is even worse. One option is to assume here for J its maximum possible value in this model which is $\nu/2$. In this case the number of states is at most $2J+1 = \nu+1$ (if one counts possible orientations of \mathbf{J}), and the entropy $\ln(\nu+1)$ is negligibly small as compared to its semiclassical value $2\pi M^2 = \nu \ln 2 / 2$. Another option is to assign here to J its semiclassical limiting value $M^2 = \nu \ln 2 / 4\pi$. But this value is much smaller numerically than ν , and in this way we effectively arrive again at formula (10), so that the extremal entropy becomes very close numerically to the Schwarzschild one instead of constituting half of it. Thus, in general case *bit is insufficient for it*.

3 Loop quantum gravity and maximum entropy principle

In a sense, the result for the quantized horizon surface derived in loop quantum gravity [8–12] can be considered as a generalization of the "it from bit" model. Here, a surface geometry is determined by a set of ν punctures on this surface (recall the patches in the "it from bit" picture). In general, each puncture is supplied by two integer or half-integer angular momenta j^u and j^d :

$$j^u, j^d = 0, 1/2, 1, 3/2, \dots \quad (13)$$

j^u and j^d are related to edges directed up and down the normal to the surface, respectively, and add up into an angular momentum j^{ud} :

$$\mathbf{j}^{ud} = \mathbf{j}^u + \mathbf{j}^d; \quad |j^u - j^d| \leq j^{ud} \leq |j^u + j^d|. \quad (14)$$

The area of a surface is

$$A = \alpha l_p^2 \sum_{i=1}^{\nu} \sqrt{2j^u(j^u+1) + 2j^d(j^d+1) - j^{ud}(j^{ud}+1)}. \quad (15)$$

A comment on the last formula is appropriate. Since the quantum numbers j entering it can be in principle arbitrarily large, the same correspondence between the power of a quantum

number and the power of \hbar (hidden here in l_p^2) in the classical limit, dictates that just the sum of square roots, but not for instance the sum of $j(j+1)$ should enter the expression for the area.

As to the overall numerical factor α in (15), it cannot be determined without an additional physical input. This ambiguity originates from a free (so-called Immirzi) parameter [13, 14] which corresponds to a family of inequivalent quantum theories, all of them being viable without such an input.

In the case of a Schwarzschild black hole, the problem was attacked in [15] by introducing into the theory boundary terms corresponding to a Chern-Simons theory on the horizon. The approach results in the situation when a single spin $j=1/2$ with two possible projections is attached to each puncture. In other words, here each radical in formula (15) contains only $j^{u(d)} = j$, $j^{d(u)} = 0$, $j^{ud} = j$, and thus reduces to $\sqrt{j(j+1)} = \sqrt{3}/2$. Obviously, in this case the entropy equals again $\nu \ln 2$, and with the Bekenstein-Hawking relation (4) one fixes immediately the numerical parameter α in formula (15): $\alpha = 2 \ln 2 / \sqrt{3}$. Further elaborations on this approach [16, 17] (see also [18]) resulted in more accurate formula (7) for the entropy of a nonrotating black hole: $S = \nu \ln 2 - 3/2 \ln \nu$. Thus, for the Schwarzschild black hole, one effectively arrives here again at the "it from bit" picture with all partial spins added up into $J = 0$. However, as to the Kerr black hole, we know already that *bit is insufficient for it*.

It is natural to investigate now whether the general Ansatz (15) is more appropriate for the description of black holes. As to the value of the overall numerical factor α in (15), one may hope that it can be determined by studying the entropy of a black hole. It has been done indeed in [19, 20] under the assumption that the entropy of an eternal black hole in equilibrium is maximum. This assumption goes back to [21], where it was used in a model of the quantum black hole as originating from dust collapse. It looks quite natural from physical point of view to assume that among the entropies of various surfaces of the given area, it is the entropy of the black hole horizon which is the maximum one.

The entropy of a surface is defined as the logarithm of the number of states of this surface with fixed area, i. e. fixed sum (15). Let ν_i be the number of punctures with a given set of momenta j_i^u, j_i^d, j_i^{ud} . The total number of punctures is

$$\nu = \sum_i \nu_i.$$

To each puncture i one ascribes a statistical weight g_i . Since $\mathbf{j}^{ud} = \mathbf{j}^u + \mathbf{j}^d$, this statistical weight equals, in the absence of other constraints, to the number of possible projections of j_i^{ud} , i. e. $g_i = 2j_i^{ud} + 1$. Then the entropy is

$$S = \ln \left[\prod_i \frac{(g_i)^{\nu_i}}{\nu_i!} \nu! \right]. \quad (16)$$

The structure of expressions (15) and (16) is so different that the proportionality between them for general values of j_i^u, j_i^d, j_i^{ud} may look impossible. However, as will be demonstrated now, this is the case for the maximum entropy in the classical limit.

But first of all let us note that the requirement of the entropy being proportional to the area is indeed quite nontrivial and restrictive. First of all, it excludes the presence of "empty" punctures, with $j_i^u = j_i^d = 0$. Such punctures would not influence the area A , would increase the combinatorics, and with it the entropy S . These "empty" punctures are excluded indeed in loop quantum gravity, but by quite different arguments.

On the other hand, this requirement forbids an excessive concentration of angular momenta at some relatively small number of punctures. The limiting case of such a concentration takes place in the black hole model suggested in [22]. Therein all angular momenta are collected effectively into a single one $J \gg 1$ at a single puncture, with the statistical weight $2J + 1$. Thus, while the area in the model [22] is $A \sim \sqrt{J(J+1)} \sim J$, the entropy is $S \sim \ln(2J+1) \sim \ln J$.

By combinatorial reasons, it is natural to expect that the absolute maximum of entropy is reached when all values of quantum numbers $j_i^{u,d,ud}$ are present. This guess is confirmed by concrete calculations for some model cases [19]. We assume also that in the classical limit the typical values of puncture numbers ν_i are large. Then, with the Stirling formula for factorials, formula (16) transforms to

$$S = \sum_i \left[\nu_i \ln g_i - \left(\nu_i + \frac{1}{2} \right) \ln \nu_i \right] + \left(\sum_i \nu_i + \frac{1}{2} \right) \times \ln \left(\sum_{i'} \nu_{i'} \right). \quad (17)$$

We have omitted here terms with $\ln \sqrt{2\pi}$, each of them being on the order of unity. The validity of this approximation, as well as of the Stirling formula by itself for this problem, will be discussed later.

We are looking for the extremum of expression (17) under the condition

$$N = \sum_i \nu_i r_i = \text{const}, \quad r_i = \sqrt{2j^u(j^u+1) + 2j^d(j^d+1) - j^{ud}(j^{ud}+1)}. \quad (18)$$

The problem reduces to the solution of the system of equations

$$\ln g_i - \ln \nu_i + \left(\sum_{i'} \ln \nu_{i'} \right) = \mu r_i, \quad (19)$$

or

$$\nu_i = g_i \exp(-\mu r_i) \sum_{i'} \nu_{i'}. \quad (20)$$

Here μ is the Lagrange multiplier for the constraining relation (18). Summing expressions (20) over i , we arrive at the equation on μ :

$$\sum_i g_i \exp(-\mu r_i) = 1. \quad (21)$$

On the other hand, when multiplying equation (19) by ν_i and summing over i , we arrive with the constraint (18) at the following result for the maximum entropy for a given value of N :

$$S_{\max} = \mu N. \quad (22)$$

Here the terms

$$-\frac{1}{2} \sum_i \ln \nu_i \quad \text{and} \quad \frac{1}{2} \ln \left(\sum_{i'} \nu_{i'} \right)$$

in expression (17) have been neglected. Below we will come back to the accuracy of this approximation.

Thus, it is the maximum entropy of a quantized surface which is proportional in the classical limit to its area. This proportionality certainly exists for a classical black hole.

And this is a very strong argument in favour of the assumption that the black hole entropy is maximum.

It should be stressed that relation (22) is true not only in the loop quantum gravity, but applies to a more general class of approaches to the quantization of surfaces. What is really necessary here, is as follows. The surface should consist of patches of different sorts, so that there are ν_i patches of a given sort i , each of them possessing a generalized effective quantum number r_i and a statistical weight g_i . Then in the classical limit, the maximum entropy of a surface is proportional to its area.

Let us consider the general case, with N given by formula (18), $g_i = 2j_i^{ud} + 1$, and all values of j_i^u, j_i^d, j_i^{ud} allowed. Here the numerical solution of equation (18) is $\mu = 3.120$, and the maximum entropy equals

$$S = 3.120N = 4.836\nu. \quad (23)$$

The mean values of quantum numbers are

$$\langle j_i^u \rangle = \langle j_i^d \rangle = 1.072, \quad \langle j_i^{ud} \rangle = 2.129. \quad (24)$$

Again, it is clear intuitively, and demonstrated explicitly by the above analysis for the simpler case of "it from bit" model, that with the assumed random distribution of j_z^{ud} over the patches the total angular momentum $\mathbf{J} = \sum \mathbf{j}_i^{ud}$ should effectively vanish. Thus, the considered model describes a Schwarzschild black hole.

It should be stressed that in this way one always arrives at the quantization rule for the black hole entropy and area effectively with integer quantum numbers ν , as was proposed in the pioneering article [2].

Let us discuss now the accuracy of our result for the maximum entropy. With all $\langle j \rangle \simeq 1$, the number of punctures ν is on the same order of magnitude as N . Thus, in the classical limit, $\nu \sim N \gg 1$. Now, according to relation (20), with $\exp(-\mu) \lesssim 1$, the numbers ν_i satisfy the condition $\nu_i > 1$ as long as the quantum numbers j are bounded by condition

$$j \lesssim \ln N. \quad (25)$$

Clearly, the typical values of those ν_i which contribute essentially to N are large, and the Stirling approximation for S is fully legitimate.

On the other hand, the number of terms in the sums in expression (17) is effectively bounded by inequality (25). Thus, the contribution of the terms with $\ln \sqrt{2\pi}$, omitted in (17), as well as of the term $1/2 \ln \nu$ retained therein, but neglected in the final expression (22), is on the order of $\ln N$ only. The leading correction to our result (22) originates from the term $-1/2 \sum \ln \nu_i$, and constitutes $\sim \ln^2 N$ in order of magnitude.

Few words on the attempts made in [23, 24] to calculate the surface entropy in loop quantum gravity. In those papers the distribution of the angular momenta j over the punctures is not discussed at all. We cannot understand how one could find the surface entropy without such information.

But what about the description of Kerr black holes in the discussed approach? It looks reasonable here to look for the maximum entropy by introducing, in line with the constraint (18) of fixed area, one more constraint, of fixed total angular momentum:

$$\sum_i j_{z,i}^{ud} = \sum_i (j_{z,i}^u + j_{z,i}^d) = J. \quad (26)$$

As to the x - and y - projections of the angular momenta, with their random distribution over the punctures, they effectively add up into zero (see the analogous discussion in the "it from bit" section).

Some progress is achieved in this way for an extremal black hole: its entropy does not vanish. However, instead of being two times smaller than the entropy of a Schwarzschild black hole of the same mass, it is smaller than the last one by a factor about 1.5 only. As to the entropy decrease for small J , it is far too small (as this was the case in the "it from bit" model). So much the more, we have not succeeded in reproducing the general relation (8).

Up to now no satisfactory solution to the problem of description of Kerr black holes in loop quantum gravity has been pointed out. This is a real challenge for the theory.

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